Name:____________________ Math 260
Start Time:_________________ Quiz 8 (35 min)
End Time:__________________
Date:______________________

1. (1, 1, 2 points) For each part below, prove or disprove that U is a subspace of $\mathbb{R}^3$:

a) $U = \{ (2s,\ s^2+1,\ -4s) \mid s \in \mathbb{R} \}$

U is <u>not</u> a subspace of $\mathbb{R}^3$ bec. $\vec{0}_{\mathbb{R}_3} = (0,0,0) \notin U$, because if $(0,0,0) \in U$, then $\exists\ s \in \mathbb{R}$ s.t. $\left.\begin{array}{l} 2s=0 \\ s^2+1=0 \\ -4s=0 \end{array}\right\} \Rightarrow$ but this system of equations has no solution bec. $2s=0 \Rightarrow s=0$, but substituting $s=0$ into s^2+1 does not make $s^2+1=0$.

b) $U = \{ (a,b,c) \mid 2a - bc = 0 \}$

U is not a subspace of $\mathbb{R}^3$ because it's not closed under addition.

$\vec{u} = (1,1,2) \in U$ bec. $2(1)-(1)(2)=0$.

$\vec{v} = (1,2,1) \in U$ bec. $2(1)-(2)(1)=0$.

But $\vec{u}+\vec{v} = (1,1,2)+(1,2,1) = (2,3,3) \notin U$ bec. $2(2)-(3)(3) = -5 \neq 0$.

(…this is a continuation of problem 1)

c) $U = \{ (a, b, c) \mid 4a + b - 3c = 0 \}$

① Is $U \neq \emptyset$? $(0,0,0) \in U$ since $4(0) + (0) - 3(0) = 0$.
So $U \neq \emptyset$.

② Is U closed under addition?

Let $\vec{u}, \vec{v} \in U$. Then $\vec{u} = (a,b,c)$ and $\vec{v} = (d,e,f)$ where $a,b,c,d,e,f \in \mathbb{R}$
and $4a + b - 3c = 0$ and $4d + e - 3f = 0$.

Then $\vec{u} + \vec{v} = (a,b,c) + (d,e,f) = (a+d, b+e, c+f) \in U$

because $4(a+d) + (b+e) - 3(c+f)$

$$= 4a + 4d + b + e - 3c - 3f$$

$$= 4a + b - 3c + 4d + e - 3f$$

$$= 0 + 0$$

$$= 0$$

③ Is U closed under scalar multiplication?

Let $\vec{v} \in U$ and let $c \in \mathbb{R}$ be a scalar.
Then $\vec{v} = (d,e,f)$ where $d,e,f \in \mathbb{R}$ and $4d + e - 3f = 0$.

Then $c\vec{v} = c(d,e,f) = (cd, ce, cf) \in U$ because

$$4(cd) + ce - 3(cf) = c\,[4d + e - 3f] = c\,[0] = 0.$$

Since $U \neq \emptyset$ and U is closed under addition
and scalar multiplication, U is a subspace (of $\mathbb{R}^3$)

2. (2, 2 points) Let $U = span(\ (1, -4, 3)\ ,\ (5, 2, -2)\ )$.

a) Is $(2, 14, -11) \in U$? If so, write (2, 14, -11) as a linear combination of (1, -4, 3) and (5, 2, -2). (Calculator OK)

Do scalars $c_1, c_2 \in \mathbb{R}$ exist such that

$c_1 (1,-4,3) + c_2 (5,2,-2) = (2,14,-11)$?

$$\left[\begin{array}{cc|c} 1 & 5 & 2 \\ -4 & 2 & 14 \\ 3 & -2 & -11 \end{array}\right] \xrightarrow{RREF} \begin{array}{cccc} c_1 & c_2 & & \\ \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

so $c_1 = -3$ and $c_2 = 1$.

$$\boxed{\begin{array}{l} \text{So } (2,14,-11) = -3(1,-4,3) + 1(5,2,-2) \\ \text{which means } (2,14,-11) \in span\{(1,-4,3),(5,2,-2)\} = U \end{array}}$$

b) Is $(13, 2, 5) \in U$? If so, write (13, 2, 5) as a linear combination of (1, -4, 3) and (5, 2, -2). (Calculator OK)

Do scalars $c_1, c_2 \in \mathbb{R}$ exist such that

$c_1 (1,-4,3) + c_2 (5,2,-2) = (13,2,5)$?

$$\left[\begin{array}{cc|c} 1 & 5 & 13 \\ -4 & 2 & 2 \\ 3 & -2 & 5 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} c_1 \quad c_2 \quad\quad \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array} \longrightarrow \text{This row means no solution.}$$

$$\boxed{\begin{array}{l} \text{No such scalars } c_1 \text{ and } c_2 \text{ exist,} \\ \text{so } (13,2,5) \notin span\{(1,-4,3),(5,2,-2)\} = U \end{array}}$$

3. (2 points) Let $U = span((4,1)\ ,\ (2,2))$. Is $U = \mathbb{R}^2$?

$\boxed{U \subseteq \mathbb{R}^2}$ since $(4,1) \in \mathbb{R}^2$ and $(2,2) \in \mathbb{R}^2$ and $\mathbb{R}^2$ is a vector space, $\mathbb{R}^2$ is closed under addition and scalar multiplication, so all linear combinations of $(4,1)$ and $(2,2)$ are in $\mathbb{R}^2$.

So $U = span\{(4,1), (2,2)\} \subseteq \mathbb{R}^2$

$\boxed{\mathbb{R}^2 \subseteq U}$ Let $(a,b) \in \mathbb{R}^2$. Is $(a,b) \in U$?

Do scalars $c_1, c_2 \in \mathbb{R}$ exist such that $c_1(4,1) + c_2(2,2) = (a,b)$?

$$\left[\begin{array}{cc|c} 4 & 2 & a \\ 1 & 2 & b \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & b \\ 4 & 2 & a \end{array}\right] \xrightarrow{-4R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & b \\ 0 & -6 & a-4b \end{array}\right]$$

$$\xrightarrow{-\frac{1}{6}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & b \\ 0 & 1 & \frac{2}{3}b - \frac{1}{6}a \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3}a - \frac{1}{3}b \\ 0 & 1 & \frac{2}{3}b - \frac{1}{6}a \end{array}\right]$$

So there is always a solution for c_1 and c_2 no matter what a and b are (bec. we didn't get all 0's in a row before the augmentation line and something nonzero after the augmentation line). Namely $c_1 = \frac{1}{3}a - \frac{1}{3}b$ and $c_2 = \frac{2}{3}b - \frac{1}{6}a$.

So $(a,b) \in span\{(4,1), (2,2)\} = U$.

Since $U \subseteq \mathbb{R}^2$ and $\mathbb{R}^2 \subseteq U \implies U = \mathbb{R}^2$ ✓